SUGGESTED SOLUTIONS TO HOMEWORK 10

Exercise 1 (9.2.3)**.** Discuss the convergence or the divergence of the series with nth term (for sufficiently large n) given by
(a) $(\ln n)^{-p}$,
(c) $(\ln n)^{-\ln n}$.

Proof. (a) We claim that $(\ln n)^{-p} > n^{-1}$ for sufficiently large n. Indeed, it suffices to note that

$$\lim_{n\to\infty} \frac{n^{-1}}{(\ln n)^{-p}} = 0.$$

Since $\sum_{n=1}^{\infty} n^{-1}$ is divergent, we have $\sum_{n=1}^{\infty} (\ln n)^{-p}$ is divergent.

(c) We claim that $(\ln n)^{-\ln n} < n^{-2}$ for sufficiently large n. Indeed, it suffices to note that

$$\lim_{n\to\infty} \frac{(\ln n)^{-\ln n}}{n^{-2}} = 0.$$

Since $\sum_{n=1}^{\infty} n^{-2}$ is convergent, we have $\sum_{n=1}^{\infty} (\ln n)^{-\ln n}$ is convergent. □

Exercise 2 (9.2.4)**.** Discuss the convergence or the divergence of the series with nth term:
(b) $n^n e^{-n}$,
(d) $(\ln n)e^{-\sqrt{n}}$.

Proof. (b) We claim that $n^n e^{-n} > 1$ for sufficiently large n. Indeed, it suffices to note that

$$\lim_{n\to\infty} n^{-n} e^{n} = 0.$$

Since $\sum_{n=1}^{\infty} 1$ is divergent, we have $\lim_{n=1}^{\infty} n^n e^{-n}$ is divergent.

(d) We claim that $(\ln n)e^{-\sqrt{n}} < n^{-2}$ for sufficiently large n, Indeed, it suffices to note that

$$\lim_{n\to\infty} \frac{(\ln n)e^{-\sqrt{n}}}{n^{-2}} = 0.$$

Since $\sum_{n=1}^{\infty} n^{-2}$ is convergent, we have $\lim_{n=1}^{\infty} (\ln n)e^{-\sqrt{n}}$ is convergent. □

Exercise 3 (9.2.7)**.** Discuss the series whose nth term is
(a) $\frac{n!}{3\cdot 5\cdot 7\cdots(2n+1)}$,
(c) $\frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}$.

Proof. (a) Since

$$\frac{n!}{3\cdot 5\cdots(2n+1)} < \frac{n!}{2\cdot 4\cdots(2n)} = 2^{-n},$$

and $\sum_{n=1}^{\infty} 2^{-n}$ is convergent, we have $\sum_{n=1}^{\infty} \frac{n!}{3\cdot 5\cdots(2n+1)}$ is convergent.

(c) Since

$$\frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)} > \frac{2\cdot 4\cdots(2n)}{4\cdot 5\cdots(2n)\cdot(2n+1)} = \frac{1}{2n+1},$$

and $\sum_{n=1}^{\infty} (2n+1)^{-1}$ is divergent, we have $\sum_{n=1}^{\infty} \frac{2\cdot 4\cdots(2n)}{3\cdot 5\cdots(2n+1)}$ is divergent. □

Exercise 4 (9.3.8)**.** Discuss the series whose nth term is:

(a) $(-1)^n \frac{n^n}{(n+1)^{n+1}}$.

(d) $\frac{(n+1)^n}{n^{n+1}}$.

Proof. (a) Since $x \ln x$ is convex, we have

$$n \ln n < \frac{1}{2}(n+1)\ln(n+1) + \frac{1}{2}(n+2)\ln(n+2),$$

which implies that

$$\frac{(n+1)^{n+1}}{(n+2)^{n+2}} < \frac{n^n}{(n+1)^{n+1}}.$$

Therefore by Alternating Series Test, we have $\sum_{n=1}^{\infty}(-1)^n \frac{n^n}{(n+1)^{n+1}}$ is convergent.

(d) Since

$$\frac{(n+1)^n}{n^{n+1}} > \frac{1}{n},$$

and $\sum_{n=1}^{\infty} n^{-1}$ is divergent, we have $\sum_{n=1}^{\infty} \frac{(n+1)^n}{n^{n+1}}$ is divergent. □

Exercise 5 (9.4.1)**.** Discuss the convergence and the uniform convergence of the series $\sum f_n$, where $f_n(x)$ is given by:

(b) $(nx)^{-2}$ $(x \neq 0)$,

(f) $(-1)^n(n+x)^{-1}$ $(x \geq 0)$.

Proof. (b) For arbitrary $a > 0$, since

$$(nx)^{-2} \leq a^{-2}n^{-2},$$

and $\sum_{n=1}^{\infty} n^{-2}$ is convergent, we have $\sum_{n=1}^{\infty}(nx)^{-2}$ is uniformly convergent for $|x| \geq a$.

However, consider $x = \pm n^{-1}$, then $f_n(\pm n^{-1}) = 1$ which implies that $\sum_{n=1}^{\infty}(nx)^{-2}$ is not uniformly convergent on $\mathbb{R}/\{0\}$.

(f) Since

$$(-1)^n(n+x)^{-1} \leq (-1)^n n^{-1},$$

and $\sum_{n=1}^{\infty}(-1)^n n^{-1}$ is convergent by Alternating Series Test, we have $\sum_{n=1}^{\infty}(-1)^n(n+x)^{-1}$ is uniformly convergent on $[0, \infty)$. □

Exercise 6 (9.4.6)**.** Determine the radius of convergence of the series $a_n x^n$, where a_n is given by:

(b) $n^{\alpha}/n!$,

(f) $n^{-\sqrt{n}}$.

Proof. (b) Since

$$\frac{n^{\alpha}}{n!}\frac{(n+1)!}{(n+1)^{\alpha}} > \frac{n+1}{2^{\alpha}},$$

which implies that the radius of convergence is ∞.

(f) Since

$$\lim_{n\to\infty} n^{-\frac{\sqrt{n}}{n}} = 1,$$

which implies that the radius of convergence is 1. □